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CONVERGENCE RATE IN STRONG LAW FOR ARRAYS OF ROWWISE AANA RANDOM VARIABLES

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Abstract. In this paper we obtain the complete convergence of weighted sums of asymptotically almost negatively associated random variables. Some previous known results for negatively associated random variables are generalized to asymptotically almost negative association case.

1. Introduction

We start introducing the concepts of negative association and asymptotically almost negative association.

A finite family of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively associated(NA) if for every pair of disjoint subsets A, B of $\{1, 2, \cdots, n\}$ and any real nondecreasing coordinatewise functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$ $Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$ whenever f and g are such that covariance exists. A infinite family of random variables is negatively associated if every finite subfamily is negatively associated([7]).

A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned} Cov(f(X_m), g(X_{m+1}, \cdots, X_{m+k})) \\ \leq q(m)(Var(f(X_m))Var(g(X_{m+1}, \cdots, X_{m+k})))^{\frac{1}{2}} \end{aligned} \tag{1.1}$$

for all $m,\ k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right hand of (1.1) is finite([2], [3]).

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The family of asymptotically almost negatively associated sequences contains negatively associated(in particular, independent) sequences(which $q(m) = 0,\ \forall m \geq 1$) and some more sequences of random variables which are not much derived from being negatively associated. An example of an asymptotically almost negatively associated sequence which is not negatively associated was constructed by Chandra and Ghosal(1996 a). Since the concept of asymptotically almost negatively associated was introduced by Chandra and Ghosal(1996 a), many applications have been found. See for example, Chandra and Ghosal(1996 a) for the Kolmogorov type inequality and the strong law of large numbers of Marcinkiewicz-Zygmund, Chandra and Ghosal(1996 b) for the almost sure convergence of weighted averages, Ko et al.(2005) for the Hájeck-Rènyi type inequality, Wang et al.(2003) for the law of the iterated logarithm for product sums, and Yuan and An(2009) for some Rosenthal type inequalities for maximum partial sums.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables. Hsu and Robbins(1947) introduced the concept of complete convergence of $\{X_n, n \geq 1\}$ as follows : A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P\{|X_n - C| \geq \epsilon\} < \infty \ for\ all\ \epsilon > 0.$$

There have been many investigations in the complete convergences: For examples, Bai and Su(1985) proved the complete convergence of partial sums of identically distributed independent random variables, Gut(1992, 1993) investigated the complete convergence of arrays of i.i.d. random variables, Li, Rao, Jiang and Wang(1995) studied complete convergence and almost sure convergence of weighted sums of independent random variables, Liang and Su(1999) and Liang(2000) derived complete convergence of weighted sums of negatively associated sequence, Kuczmaszewska(2009) showed the complete convergence for arrays of rowwise negatively associated random variables, Wu(2010) studied the complete convergence for negatively dependent sequences of random variables.

2. Some lemmas

Definition 2.1. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variables X if there exists a constant C such that $P(|X_n| \geq x) \leq CP(|X| \geq x)$ for all $x \geq 0,\ n \geq 1$.

The following lemma is well known result:

LEMMA 2.2. *([13] Lemma 2.2) Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variables X. Then for any $p > 0$ and $x > 0$*

$$E|X_n|^p I(|X_n| < x) \leq C\{|X|^p I(|X| < x) + x^p P(|X| \geq x)\}, \tag{2.1}$$

$$E|X_n|^p I(|X_n| \geq x) \leq CE\{|X|^p I(|X| \geq x)\}. \tag{2.2}$$

LEMMA 2.3. *Let $\{X_i, 1 \leq i \leq n\}$ be a finite family of asymptotically almost negatively associated random variables with $\{q(n), n \geq 1\}$ such that $q(n) \to 0$ as $n \to \infty$. Let for any real numbers a_i and b_i such that $a_i < b_i$ for $1 \leq i \leq n$,*

$$Y_i = b_i I[X_i > b_i] + X_i I[a_i \leq X_i \leq b_i] + a_i I[X_i < a_i].$$

Then $\{Y_i, 1 \leq i \leq n\}$ is still asymptotically almost negatively associated.

LEMMA 2.4. *([15] Lemma 2.4) Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of asymptotically almost negatively associated random variables with mean zero and $p \in (3 \cdot 2^{k-1}, 4 \cdot 2^{k-1})$, for every positive integer k. Then there exists a positive $\mathcal{D}_p$ such that*

$$E \max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i|^p \leq \mathcal{D}_p\{\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{\frac{p}{2}}\}. \tag{2.3}$$

3. Main results

In spired by Kuczmaszewska(2009) we obtain the following complete convergence for asymptotically almost negative associated random variables.

THEOREM 3.1. *Let $\{X_i, i \geq 1\}$ be a sequence of asymptotically almost negatively associated random variables and $\{a_{ni}, n \geq 1\}$ be an array of real numbers. Let $\{c_n, n \geq 1\}$ be a sequence of positive numbers. Assume that*

$$\sum_{i=1}^{n} P\{|a_{ni}X_i| \geq n^\alpha\} < \infty, \tag{a}$$

$$\sum_{n=1}^{\infty} c_n n^{-2\alpha} \sum_{i=1}^{n} |a_{ni}|^2 E(|X_i|^2 I[|a_{ni}X_i| < n^\alpha) < \infty. \tag{b}$$

Then for all $\epsilon > 0$

$$\sum_{n=1}^{\infty} c_n P\{\max_{1\le k\le n} |\sum_{i=1}^{k}(a_{ni}X_i - a_{ni}EX_iI[|a_{ni}X_i| < n^\alpha])| > \epsilon n^\alpha\} < \infty. \tag{3.1}$$

Proof. If the series $\sum_{n=1}^{\infty} c_n < \infty$, then (3.1) always holds. So we consider only the case that $\sum_{n=1}^{\infty} c_n$ is divergent. Let $a_{ni}X_i^{'} = a_{ni}X_iI[|a_{ni}X_i| < n^\alpha] + n^\alpha I[a_{ni}X_i \ge n^\alpha] - n^\alpha I[a_{ni}X_i \le -n^\alpha]$ and $T_k = \sum_{i=1}^{k}(X_i^{'} - EX_i^{'})$. Then $\{T_k, k \ge 1\}$ is also a sequence of asymptotically almost negatively associated random variables by Lemma 2.3.

By (a), for sufficient large n we observe

$$\begin{aligned} &P\{\max_{1\le k\le n^\alpha} |\sum_{i=1}^{k}(a_{ni}X_i - a_{ni}E(X_iI[|a_{ni}X_i| < n^\alpha]))| > \epsilon n^\alpha\} \\ &\le C[\sum_{i=1}^{n^\alpha} P\{|a_{ni}X_i| \ge n^\alpha\} + \epsilon^{-2}n^{-2\alpha}E(\max_{1\le i\le n^\alpha} |T_i|^2)]. \end{aligned}$$

Using the C_r inequality, we can estimate

$$E(a_{ni}|X_i^{'} - EX_i^{'}|^r) \le C(E(|a_{ni}X_i|^rI[|a_{ni}X_i| < n^\alpha]) + n^{\alpha r}P\{|a_{ni}X_i| \ge n^\alpha\}).$$

Thus by the above estimates and Lemma 2.4 we have

$$\begin{aligned} &P\{\max_{1\le k\le n^\alpha} |\sum_{i=1}^{k}(a_{ni}X_i - E(a_{ni}X_iI[|a_{ni}X_i| < n^\alpha]))| > n^\alpha\} \\ &\le C[\sum_{i=1}^{n^\alpha} P\{|a_{ni}X_i| \ge n^\alpha\} + n^{-2\alpha}(\sum_{i=1}^{n^\alpha} a_{ni}^2EX_iI[|a_{ni}X_i| < n^\alpha])]. \end{aligned} \tag{3.2}$$

Therefore from (a), (b) and (3.2) we obtain that (3.1) holds. The proof of Theorem 3.1 completes. □

THEOREM 3.2. *Let* $\{X_i, i \ge 1\}$ *be a sequence of asymptotically almost negatively associated random variables and* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of real numbers. Let* $\{c_n, n \ge 1\}$ *be a sequence of positive numbers. Assume that (a) and (b) in Theorem 3.1 and*

$$\max_{j\ge n} |\sum_{i\le j}(Ea_{ni}X_iI[|a_{ni}X_i| \le n^\alpha])| = o(n^\alpha) \tag{c}$$

hold. Then we have

$$\sum_{n=1}^{\infty} c_n P\{\max_{1\le i\le n^\alpha} |\sum_{j=1}^{i}(a_{nj}X_j)| > \epsilon n^\alpha\} < \infty \ for\ all\ \epsilon > 0. \tag{3.3}$$

COROLLARY 3.3. *Let $\{X_i, i \ge 1\}$ be a sequence of asymptotically almost negatively associated random variables such that $EX_i = 0$ for all $i \ge 1$ and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers. Let $\{c_n, n \ge 1\}$ be a sequence of positive numbers. Assume that for some sequence $\{\lambda_n, n \ge 1\}$ with $0 < \lambda_n \le 1$ such that $E|X_i|^{1+\lambda_n} < \infty$. If*

$$\sum_{n=1}^{\infty} c_n (n^\alpha)^{-1-\lambda_n} \sum_{i=1}^{n^\alpha} E|a_{ni}X_{ni}|^{1+\lambda_n} < \infty, \tag{3.4}$$

then for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} c_n P\{\max_{1\le i\le n^\alpha} |\sum_{j=1}^{i} a_{nj}X_j| > \epsilon n^\alpha\} < \infty. \tag{3.5}$$

Proof. We note that it is enough to prove the case when $\sum_{n=1}^{\infty} c_n$ is infinite, because in the case when $\sum_{n=1}^{\infty} c_n$ is finite (3.5) always holds.

In the case when $\sum_{n=1}^{\infty} c_n$ is divergent we see that (3.4) implies

$$(n^\alpha)^{-1-\lambda_n} \sum_{i=1}^{n^\alpha} |a_{ni}|^{1+\lambda_n} E|X_i|^{1+\lambda_n} < 1. \tag{3.6}$$

By assumption (3.4) we immediately obtain that conditions (a) and (b) in Theorem 3.1 hold. Indeed

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \sum_{i=1}^{n^\alpha} P\{|a_{ni}X_i| > \epsilon n^\alpha\} \\
&< \sum_{n=1}^{\infty} c_n (n^\alpha)^{-1-\lambda_n} \sum_{i=1}^{n^\alpha} |a_{ni}|^{1+\lambda_n} E|X_i|^{1+\lambda_n} < \infty
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n n^{-2\alpha} (\sum_{i=1}^{n^\alpha} a_{ni}^2 EX_i^2 I[|a_{ni}X_i| < \epsilon n^\alpha]) \\
&\le \sum_{n=1}^{\infty} c_n (n^\alpha)^{-1-\lambda_n} (\sum_{i=1}^{n^\alpha} |a_{ni}|^{1+\lambda_n} E|X_i|^{1+\lambda_n}) < \infty.
\end{aligned}$$

To complete the proof, it is enough to show that

$$n^{-\alpha} \max_{1 \le i \le n^{\alpha}} |\sum_{j=1}^{i} a_{nj} E X_j I[|a_{nj} X_j| < \epsilon n^{\alpha}]| \to 0 \ as \ n \to \infty.$$

Since $EX_i = 0$ we have

$$\begin{aligned} & n^{-\alpha} \max_{1 \le i \le n^{\alpha}} |\sum_{j=1}^{i} a_{nj} E X_j I[|a_{nj} X_j| < \epsilon n^{\alpha}]| \\ & \le n^{-\alpha} \max_{1 \le i \le n^{\alpha}} \sum_{j=1}^{i} |a_{nj} E X_j I[|a_{nj} X_j| > \epsilon n^{\alpha}]| \\ & = (n^{\alpha})^{-(1+\lambda_n)} \sum_{j=1}^{n^{\alpha}} |a_{nj}|^{1+\lambda_n} E|X_j|^{1+\lambda_n} \to 0 \ as \ n \to \infty. \end{aligned}$$

The proof is completed. □

COROLLARY 3.4. *Let $\{X_i, i \ge 1\}$ be a sequence of asymptotically almost negatively associated random variables such that $EX_i = 0$ and $E|X_i|^p < \infty$ for $i \ge 1$ and $1 < p \le 2$. Let $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers satisfying condition*

$$\sum_{i=1}^{n} |a_{ni}|^p E|X_i|^p = O(n^{\delta}) \ as \ n \to \infty \tag{3.7}$$

for some $0 < \delta \le 1$. Then for any $\epsilon > 0$ and $\alpha p \ge 1$

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\{\max_{1 \le i \le n} |\sum_{j=1}^{i} a_{nj} X_j| > \epsilon n^{\alpha}\} < \infty \tag{3.8}$$

Proof. Put $c_n = n^{\alpha p - 2}$ in Theorem 3.1. Then by (3.7) we have

$$\begin{aligned} \sum_{n=1}^{\infty} c_n \sum_{i=1}^{n^{\alpha}} P\{|a_{ni} X_i| \ge \epsilon n^{\alpha}\} &< \epsilon^{-p} \sum_{n=1}^{\infty} n^{\alpha p - 2} \sum_{i=1}^{n} \frac{|a_{ni}|^p E|X_i|^p}{n^{\alpha p}} \\ &\le C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty, \end{aligned}$$

and

$$\sum_{n=1}^{\infty} c_n n^{-2\alpha} \sum_{i=1}^{n^\alpha} a_{ni}^2 E(X_i^2 I[|a_{ni}X_i| < \epsilon n^\alpha])$$
$$\leq \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}|^p E|X_i|^p \leq C \sum_{n=1}^{\infty} n^{-2+\delta} < \infty.$$

To complete the proof, it is enough to show that by assumption $EX_i = 0$ for $i \geq 1$ and by (3.7) we get

$$\frac{1}{n^\alpha} \sum_{j=1}^{i} a_{nj} E(X_j I[|a_{nj}X_j| < \epsilon n^\alpha]) \to 0 \ as \ n \to \infty \ for \ 1 \leq i \leq n.$$

□

COROLLARY 3.5. *Let $\{X_i, i \geq 1\}$ be a sequence of asymptotically almost negatively associated random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for $i \geq 1$ and $1 < p \leq 2$. Let the random variables X_i be stochastically dominated by a random variable X such that $E|X|^p < \infty$. If $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying the condition*

$$\sum_{i=1}^{n} |a_{ni}|^p = O(n^\delta) \ as \ n \to \infty,$$

for some $0 < \delta \leq 1$. Then for any $\epsilon > 0$ and $\alpha p \geq 1$ (3.8) holds.

COROLLARY 3.6. *Let $\{X_i, i \geq 1\}$ be a sequence of asymptotically almost negatively associated random variables with $EX_i = 0$ and the random variables X_i be stochastically dominated by a random variable X. Let a double array $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ of real numbers satisfy*

$$(i) \qquad \lim_{n\to\infty} a_{ni} = 0 \ for \ all \ i \geq 1,$$

$$(ii) \qquad \sum_{i=1}^{\infty} |a_{ni}| \leq C \ for \ all \ n \geq 1,$$

where C is a positive constant. If for some $0 < \delta \leq 1$

$$(3.9) \qquad \sup_{i\geq 1} |a_{ni}| = O(n^{\alpha-\delta}) \ and \ E|X|^{1+\frac{1}{\delta}} < \infty,$$

then for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{\max_{1\leq i\leq n} |\sum_{j=1}^{i} a_{nj}X_j| > \epsilon n^\alpha\} < \infty.$$

Proof. Let $c_n = 1$.

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{n} P\{|a_{ni}X_i| \geq \epsilon n^{\alpha}\} \leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\{|a_{ni}X| \geq \frac{\epsilon n^{\alpha}}{\mathcal{D}}\}$$
$$\leq \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\{|X| \geq C\epsilon n^{\delta}\} \leq \sum_{n=1}^{\infty} nP\{C\epsilon n^{\delta} \leq |X| < C\epsilon(n+1)^{\delta}\}$$
$$\leq CE|X|^{\frac{1}{\delta}} < \infty.$$

$$\sum_{n=1}^{\infty} c_n n^{-2\alpha} \sum_{i=1}^{n} a_{ni}^2 EX_i^2 I[|a_{ni}X_i| < \epsilon n^{\alpha}]$$
$$\leq C\sum_{n=1}^{\infty} n^{-2\alpha} \sum_{i=1}^{n} a_{ni}^2 (E(X^2 I[|a_{ni}X_i| < \epsilon n^{\alpha}]) + \frac{n^{2\alpha}}{a_{ni}^2} P\{|a_{ni}X| \geq \epsilon n^{\alpha}\})$$
$$\leq C\sum_{n=1}^{\infty} n^{-\alpha(1+\frac{1}{\delta})} \sum_{i=1}^{n} |a_{ni}|^{1+\frac{1}{\delta}} E|X|^{1+\frac{1}{\delta}} + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\{|a_{ni}X| \geq \epsilon n^{\alpha}\}$$
$$\leq C\sum_{n=1}^{\infty} n^{-(\alpha+1)} E|X|^{1+\frac{1}{\delta}} \sum_{i=1}^{n} |a_{ni}| + CE|X|^{\frac{1}{\delta}}$$
$$\leq C\sum_{n=1}^{\infty} n^{-\alpha-1} + CE|X|^{\frac{1}{\delta}} < \infty.$$

To complete the proof we should prove $|n^{-\alpha}\sum_{j=1}^{i} a_{nj}EX_jI[|a_{nj}X_j| < \epsilon n^{\alpha}]| \to 0 \ as \ n \to \infty$ for each $1 \leq i \leq n$. We have

$$|n^{-\alpha}\sum_{j=1}^{i} a_{nj}EX_jI[|a_{nj}X_j| < \epsilon n^{\alpha}]|$$
$$\leq C(n^{-\alpha}\sum_{j=1}^{i} |a_{nj}|E|X| + \sum_{j=1}^{n} P\{|a_{nj}X| \geq \epsilon n^{\alpha}\})$$
$$\leq Cn^{-\alpha}E|X|(\sum_{j=1}^{i} |a_{nj}|) + \sum_{j=1}^{n} P\{|a_{nj}X| \geq \epsilon n^{\alpha}\}$$
$$\leq C[n^{-\alpha} + \sum_{j=1}^{n} P\{|a_{nj}X| > \epsilon n^{\alpha}\}] \to 0 \ as \ n \to \infty.$$

Thus the proof is completed. □

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